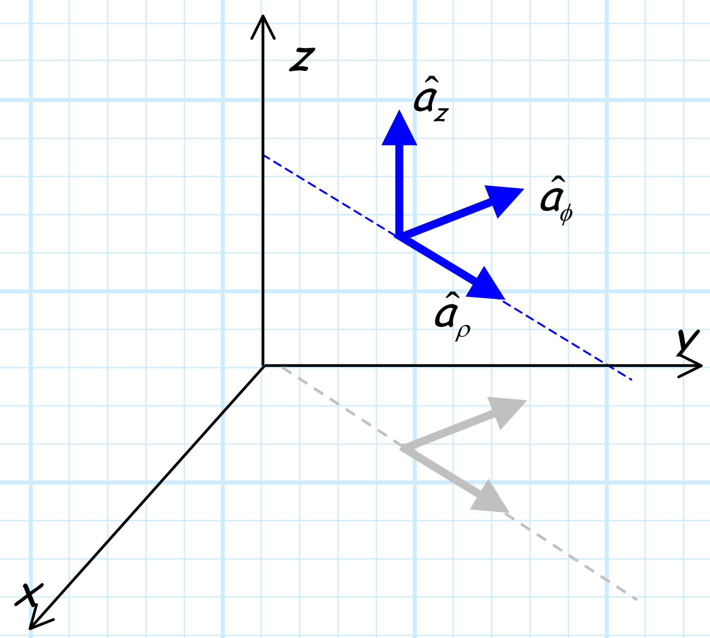
Cylindrical Base Vectors

Cylindrical base vectors are the **natural** base vectors of a **cylinder**.

$\hat{a}_\rho$ points in the direction of **increasing** ρ. In other words, $\hat{a}_\rho$ points **away from the *z*-axis**.

$\hat{a}_\phi$ points in the direction of **increasing** ϕ. This is precisely the **same** base vector we described for **spherical** base vectors.

$\hat{a}_z$ points in the direction of **increasing *z***. This is precisely the **same** base vector we described for **Cartesian** base vectors.



It is evident, that like spherical base vectors, the cylindrical base vectors are **dependent on position**. A vector that points **away** from the z-axis (e.g., $\hat{a}_\rho$), will point in a direction that is **dependent** on where we are in space!

We can express cylindrical base vectors in terms of **Cartesian** base vectors. First, we find that:

$$\begin{array}{lll}
\hat{a}_\rho \cdot \hat{a}_x = \cos\phi & \hat{a}_\phi \cdot \hat{a}_x = -\sin\phi & \hat{a}_z \cdot \hat{a}_x = 0 \\
\hat{a}_\rho \cdot \hat{a}_y = \sin\phi & \hat{a}_\phi \cdot \hat{a}_y = \cos\phi & \hat{a}_z \cdot \hat{a}_y = 0 \\
\hat{a}_\rho \cdot \hat{a}_z = 0 & \hat{a}_\phi \cdot \hat{a}_z = 0 & \hat{a}_z \cdot \hat{a}_z = 1
\end{array}$$

We can use these results to write **cylindrical** base vectors in terms of **Cartesian** base vectors, or vice versa!

For **example**,

$$\begin{aligned}
\hat{a}_p &= \left(\hat{a}_p \cdot \hat{a}_x\right)\hat{a}_x + \left(\hat{a}_p \cdot \hat{a}_y\right)\hat{a}_y + \left(\hat{a}_p \cdot \hat{a}_z\right)\hat{a}_z \\
&= \cos\phi\, \hat{a}_x + \sin\phi\, \hat{a}_y
\end{aligned}$$

or,

$$\begin{aligned}
\hat{a}_x &= \left(\hat{a}_x \cdot \hat{a}_\rho\right)\hat{a}_\rho + \left(\hat{a}_x \cdot \hat{a}_\phi\right)\hat{a}_\phi + \left(\hat{a}_x \cdot \hat{a}_z\right)\hat{a}_z \\
&= \cos\phi\, \hat{a}_\rho - \sin\phi\, \hat{a}_\phi
\end{aligned}$$

Finally, we can write **cylindrical** base vectors in terms of **spherical** base vectors, or vice versa, using the following relationships:

$$\hat{a}_\rho \cdot \hat{a}_r = \sin\theta \qquad \hat{a}_\phi \cdot \hat{a}_r = 0 \qquad \hat{a}_z \cdot \hat{a}_r = \cos\theta$$

$$\hat{a}_\rho \cdot \hat{a}_\theta = \cos\theta \qquad \hat{a}_\phi \cdot \hat{a}_\theta = 0 \qquad \hat{a}_z \cdot \hat{a}_\theta = -\sin\theta$$

$$\hat{a}_\rho \cdot \hat{a}_\phi = 0 \qquad \hat{a}_\phi \cdot \hat{a}_\phi = 1 \qquad \hat{a}_z \cdot \hat{a}_\phi = 0$$

e.g.,

$$\begin{aligned} \hat{a}_p &= \left(\hat{a}_p \cdot \hat{a}_r \right) \hat{a}_r + \left(\hat{a}_p \cdot \hat{a}_\theta \right) \hat{a}_\theta + \left(\hat{a}_p \cdot \hat{a}_\phi \right) \hat{a}_\phi \\ &= \sin\theta \, \hat{a}_r + \cos\theta \, \hat{a}_\theta \end{aligned}$$

$$\begin{aligned} \hat{a}_\theta &= \left(\hat{a}_\theta \cdot \hat{a}_\rho \right) \hat{a}_\rho + \left(\hat{a}_\theta \cdot \hat{a}_\phi \right) \hat{a}_\phi + \left(\hat{a}_\theta \cdot \hat{a}_z \right) \hat{a}_z \\ &= \cos\theta \, \hat{a}_\rho - \sin\theta \, \hat{a}_z \end{aligned}$$